

The number $e^{\frac{1}{2}}$ is the ratio between the time of maximum value and the time of maximum growth rate for restricted growth phenomena?

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Abstract

For many natural process of growth, with the growth rate independent of size due to Gibrat law and with the growth process following a log-normal distribution, if the growth process is restricted so that the production of Shannon entropy is maximized, the ratio between the time (D) for maximum value and the time (L) for maximum growth rate (inflexion point) is then equal to the square root of the base of the natural logarithm ($e^{1/2}$). On the logarithm scale this ratio becomes one half ($\frac{1}{2}$). It remains an open question, due to lack of complete data for various cases with restricted growth, whether this $e^{1/2}$ ratio can be stated as $e^{1/2}$ -Law. Two established examples already published, one for an epidemic spreading and one for droplet production, support however this ratio. Another rough example appears to be the height of human body. For boys the maximum height occurs near 23 years old while the maximum growth rate is at the age near 14, and there ratio is close to $e^{-1/2}$.

Keywords. Growth phenomena, maximum, maximum growth rate, universal ratio, the number e .

1 Natural growth phenomena, $e^{1/2}$ law and discussion

For many natural phenomena with restricted growth process, the number $f(t)$ in growth first increases, reaching to its maximum growth rate at some time (denoted $t = L$), and finally decays after the maximum occurs (at time $t = D$). Examples include measures of size of living tissues (length, skin area and weight) and spreading of epidemics in biology, restricted population growth and income distribution in social science, expansion of city size, spray process in technology, [1] etc.

Here, together with the analysis and data of two previous papers [2]-[3] where however the generality and simplicity of the result were not revealed (at least the number e is not found in the formulas published there), we want to show, though we prefer not to use prove, that the ratio between the time for maximum and the time for maximum growth rate satisfies the following $e^{\frac{1}{2}}$ law

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$$\frac{D}{L} = e^{\frac{1}{2}} \text{ or } \frac{D^2}{L^2} = e \text{ or } \ln \frac{D}{L} = \frac{1}{2} \text{ (} e^{\frac{1}{2}} \text{ law)} \quad (1.1)$$

Below are important discussions.

Simplicity and generality. The ratio expression (1.1) is only related to e , the natural exponent

$$e = 2.7182 \dots$$

This is elegant and simple. Does this add to the mystery [4]-[5] of this number? On the log scale this ratio becomes one half, the simplicity of which deserves further imagination. Most importantly, the two characteristic time scales L and D should be problem dependent according to our common sense, while according to (1.1) their ratio is problem independent, without any other free parameters needing adjustment. Does this mean that simplicity is always associated with complexity [6]?

Simple way to obtain the results. In references [2]-[3], this problem has been studied for spreading of droplet size and for spreading of epidemics. Though the basic method for analysis and for obtaining the results has been reported in [2]-[3], there is no remark about the generality, notably, the ratio L/D is not considered in [2] while it is expressed as $L/D = 1.649$ in [3], without noticing that it can be expressed in the elegant form of (1.1). In the next section we shall repeat the analysis to obtain (1.1), as if for a more general problem, rather than just for spray process as in [2], or epidemic spreading as in [3]. Moreover, we shall use more experimental data to support the intermediate theoretical parameters during the derivation for (1.1). The basic idea is to assume that $f(t)$, which can be expressed as the log-normal distribution, is associated with an entropy, and for restricted growth problem, the way for restriction is such that the rate of production of entropy is maximized.

Validation and agreement. It is remarkable that the SARS data in the years of 2003 obeys $L/D = 1.649$ thus (1.1), according to reference [3]. The reported hospitalized cases for each city obeys this rule individually. Though this ratio is not considered for droplet production in reference [2], the agreement between the experimental data and the theoretical curve $f(t)$ mean that (1.1) is also satisfied, since (1.1) is implicated in the theoretical curve $f(t)$. Another rough example appear to be the height of human body. For boys the maximum height occurs near 23 years old while the maximum growth rate is at the age near 14. If we use these data (the data could not be exact, and sometimes a range is given, see [7]), the ratio is close to $e^{-1/2}$. Validations against other types of growth problem are necessary to assess further the generality of (1.1). However, data were not reported as we require and we have to leave further comparisons to those we have accumulated the data for the specific problems they consider. It is interesting to know counter examples. The spreading of droplet size and the spreading of SARS are however quite different problems and the fact that they obey (1.1) would mean the possibility of generality.

How to use (1.1). It would be just for enjoy to see if for a production process the law (1.1) is obeyed. For instance the growth of height of human body happens to comply with it roughly. One may wonder how it is possible that this law is independent of problem. However, if this law is incorrect, the departure between it and the observation of the above three types of examples would be extremely great! If this could be further assessed by many other examples, then we may wish to use it to predict the (restricted) growth, notably the time D for maximum, after the inflexion point (L) appears. However, even this is indeed so, there is a problem to know the initial time for production (that is L , that should be counted from the first time production occurs, is unknown even though we know the day at which the maximum growth rate occurs). For instance, if this is an epidemic, then it is very hard to know the first day that the epidemic

starts. This, however, could be evaluated as follows. At the inflexion point, we record the number $f(L)$ and its growth rate $df(L)dt$, then L can be computed as (see next section)

$$L = \frac{3f(L)}{\left. \frac{df(t)}{dt} \right|_{t=L}} \quad (1.2)$$

2 The way to obtain the results

As remarked above, most materials have appeared in references [2]-[3], though not considered as a general way and the result is not expressed in its elegant form (1.1). Here we consider the problem to be universal, repeat the essential analysis, reexpress the results in its form as stated in section 1, and use more data to support the intermediate parameter required to achieve finally (1.1) and (1.2).

Many natural phenomena in growth can be described by the log-normal distribution [1],

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma t} \exp\left(-\frac{(\ln t - \mu)^2}{2\sigma^2}\right) \quad (2.1)$$

since the growth rate is independent of size following the Gibrat's law[8]. For spreading of epidemics, and in fact the procedure is not restricted to epidemics, it was proved that (2.1) is the solution of a very simple equation for growth rate[3]. Here $\mu = \ln D + \sigma^2$ is a location parameter and the standard deviation σ is the scale parameter on the logarithmic scale. For a particular process, σ is often fitted against experimental data in the literature, without remarking whether there be some intrinsic mechanism so that σ must take some specific value.

In fact, the standard deviation σ characterizes the width of $f(t)$. The internal growth mechanism would make the curve $f(t)$ wider and wider. If the growth process is unrestricted, then $f(t)$ will continue to grow without a maximum. For instance, for a highly communicable epidemic such as SARS, if there is no restriction such as publication intervention, then the majority of population will be infected. If there is no restriction, then a city size will continue to grow. The restriction effect provides a dissipation mechanism to prevent the curve to become infinitely wide. The width cesses to increase when the maximum dissipation rate is reached. According to reference [9], maximum dissipation rate corresponds to maximum entropy production. This is the so-called maximum entropy production principle, though questioned as a general principle but often found correct and useful for determining parameters.

Use the Shannon entropy $S(\sigma) = -\int_0^\infty t^{1-3} f(t) \ln(t^{1-3} f(t)) dt^3$. It is found that [2] for $f(t)$ defined by (2.1),

$$S(\sigma) = 3 \left(\ln(\sqrt{2\pi}\sigma) + 3(\ln D + \sigma^2) + \frac{1}{2} \right)$$

Maximizing the entropy production rate, that is setting $\frac{d^2 S(\sigma)}{d\sigma^2} = 0$, yields

$$\sigma = \frac{\sqrt{6}}{6} \quad (2.2)$$

In reference [3], it is the value $\sigma = 0.401$ that is used. For spray processs, references [10],[11] and [12] fitted the experimental data to give the following range of σ ,

$$\sigma = (0.977, 1.71) \frac{\sqrt{6}}{6}, \quad \sigma = (1.009, 1.065) \frac{\sqrt{6}}{6} \text{ and } \sigma = 1.1023 \frac{\sqrt{6}}{6} \quad (2.3)$$

The comparison in references [2]-[3] with experimental data and the close agreement between (2.3) and (2.2) support the theoretical value of (2.2).

Now for $f(t)$ defined by (2.1), we have

$$\frac{d^2 f(t)}{dt^2} = g(t) \left(\sigma^2 \ln \frac{t}{D} - \sigma^2 + \ln^2 \frac{t}{D} \right) \quad (2.4)$$

where

$$g(t) = \frac{1}{2} \frac{\sqrt{2}}{\sqrt{\pi} t^3 \sigma^5} \exp \left(-\frac{1}{2\sigma^2} \left(\ln \frac{t}{D} - \sigma^2 \right)^2 \right) > 0$$

By definition of inflexion point, i.e., $t = L$ at which $\frac{d^2 f(L)}{dt^2} = 0$, we obtain from (2.4) the following relation

$$\sigma^2 \ln \frac{L}{D} - \sigma^2 + \ln^2 \frac{L}{D} = 0$$

which can be solved to give

$$\frac{D}{L} = e^{\frac{1}{2}} \text{ or } \frac{D^2}{L^2} = e \quad (2.5)$$

where $e = 2.71828 \dots$ is the natural exponential!

To determine the initial date for regular production, or to know the number L (cumulated days to reach the inflexion point counting from the initial date), we use (2.1) to write

$$\left. \frac{df(t)}{dt} \right|_{t=L} = -\frac{1}{\sigma^2 L} f(L) \ln \frac{L}{D}$$

Hence

$$L = \left(\frac{1}{2} + \frac{1}{2} \sqrt{\frac{4}{\sigma^2} + 1} \right) \frac{f(L)}{\left. \frac{df(t)}{dt} \right|_{t=L}} = \frac{3f(L)}{\left. \frac{df(t)}{dt} \right|_{t=L}} \quad (2.6)$$

Hence we have given all the required derivation for what is needed in section 1.

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